Integrability of two-dimensional homogeneous potentials

1988 J. Phys. A: Math. Gen. 21 2291

(http://iopscience.iop.org/0305-4470/21/10/009)

J. Phys. A: Math. Gen. 21 (1988) 2291-2299. Printed in the UK

Integrability of two-dimensional homogeneous potentials

M P Joy and M Sabir

Department of Physics, Cochin University of Science and Technology, Cochin-682022, India

Received 10 November 1987, in final form 14 January 1988

Abstract. Combining the extended Painlevé conjecture with Yoshida's singularity and stability analyses it is shown that, for two-dimensional homogeneous potentials of degree $2m$, integrability restricts Kowalevskaya exponents and integrability coefficients to discrete sets of values. This result is made use of in the analysis of integrability of symmetric potentials with $m=2$, 3 and 4. Direct construction of additional first integrals is successful only in special cases which can be transformed to known integrable ones.

1. Introduction

In recent literature considerable attention has been paid to the question of integrability of Hamiltonian dynamical systems. A Hamiltonian system of N degrees of freedom is said to be integrable if there exist N analytic, single-valued time-independent first integrals in involution. Such systems possess several remarkable properties and are of great importance from practical as well as theoretical points of view. However, given a Hamiltonian system it is not possible to say whether it is integrable or not except when one can construct first integrals directly. Even though such constructions can be carried out to a certain extent, particularly for low-dimensional systems, the results are neither exhaustive nor conclusive (Hietarinta 1987). In recent times it has been realised that singularity analysis and stability analysis can shed considerable light on the question of integrability. The Painlevé analysis (Ablowitz *et al* 1980) and Kowalevskaya analysis of Yoshida (1983) are two approaches to singularity analysis. Roekaerts and Schwarz (1987) have shown that these methods can be combined to obtain stronger conditions on integrability. Yoshida (1984, 1986) has also shown that the stability of certain types of solutions is directly linked with the existence of first integrals. A number of candidates for integrable systems have been identified by these methods and their combinations.

In this paper we analyse systems with Hamiltonian of the form

$$H=\tfrac{1}{2}(P_x^2+P_y^2)+V(x,y) \tag{1}$$

with $V(x, y)$ a homogeneous polynomial potential of even degree $2m$. Hamiltonians of these types are used in lattice dynamics, condensed matter physics, field theory, astrophysics, etc, and special cases of these have been studied in the existing literature (Bountis *et al* 1982, Dorizzi *et al* 1983, Grammaticos *et al* 1983, Steeb *et al* 1985). By combining singularity and stability analyses we have obtained a stronger condition for integrability as a restriction on the possible Kowalevskaya exponents (KE) and integrability coefficients. We have also carried out singularity and stability analyses of symmetric homogeneous potentials with $m = 2$, 3 and 4 to identify possible integrable cases. A second integral is also constructed directly in those cases suggested by these analyses. We have generalised the integrable cases to a potential of arbitrary degree $2m$ by constructing the corresponding second integral.

In § 2, we present a brief summary of the singularity and stability analyses and their implications concerning integrability of a system. Here is also presented our results concerning the restrictions on KE obtained by combining singularity and stability analyses. The results of the analyses carried out for quartic, sextic and octic potentials are presented in § 3. Second integrals for the identified integrable cases are also given here. Section 4 summarises our conclusions.

2. Singularity, stability and integrability

2.1. *Painlevé analysis*

According to the extended Painlevé conjecture (Ramani *et al* 1982) a sufficient condition for integrability is the weak Painlevé property (WPP). A system of equations is said to have the strong Painlevé property (PP) when the only movable singularities of the solutions in the complex time plane are poles. In the weak Painlevé case certain algebraic branch points are also allowed. A strong necessary condition for the PP or WPP is provided by the Painlevé analysis (P-analysis) (Ablowitz *et al* 1980, Graham *et al* 1985).

In P-analysis we try to find solutions around a movable singularity at t_0 in the complex time plane in the form

$$\begin{aligned} x(t) &= \sum_{j=0}^{\infty} a_j \tau^{-p+j/s} \\ y(t) &= \sum_{j=0}^{\infty} b_j \tau^{-q+j/s} \end{aligned} \tag{2}$$

where $\tau = t - t_0$ and p and q are positive rational numbers with a common integer denominator $s > 0$. $s \neq 1$ corresponds to WPP. There are three steps in the P-analysis.

Step 1. Find the dominant behaviour. When

$$x(t) = a_0 \tau^{-p} \qquad \text{and} \qquad y(t) = b_0 \tau^{-q} \tag{3}$$

are inserted in the equations of motion, for certain values of p and q some terms of the equations may balance while others can be ignored for $t \to t_0$. These terms are called dominant terms.

Step 2. Find the resonances. To find the resonances r, substitute

$$\begin{aligned} x &= a_0 \tau^{-p} + c\tau^{-p+r} \\ y &= b_0 \tau^{-q} + d\tau^{-q+r} \end{aligned} \tag{4}$$

in the linearised form of the dominant terms. Resonances are those values of r for which the determinant of the linear system satisfied by ${}^{\mathrm{T}}(c, d)$ vanishes. For the system to have PP, $j = rs$ must be integers.

Step 3. Find the constants of integration. It is tested whether the positive resonances do indeed correspond to free parameters in a solution to the full equations of motion without logarithmic singularities. This is done by using expansion of solutions (2) up to the largest value of the resonance.

If the system passes all the three steps we say that it is a P-case.

2.2. *Kowalevskaya exponents*

Hamiltonians with homogeneous potentials of degree $2m$ are invariant under the similarity transformation

$$\begin{array}{lll} t\to\alpha^{-1}t & x\to\alpha^{g}x & y\to\alpha^{g}y \\ & P_x\to\alpha^{g'}P_x & P_y\to\alpha^{g'}P_y \end{array} \tag{5}$$

where $g=1/(m-1)$ and $g'=m/(m-1)$. If under the above transformation a function (polynomial) gets multiplied by α^M it is said to be of weighted degree M. Using the constants k_1 and k_2 as determined from the equations,

$$gg'k_1=\frac{-\partial H}{\partial x}(k_1 k_2) \qquad gg'k_2=\frac{-\partial H}{\partial y}(k_1,k_2) \tag{6}$$

we define a 4×4 matrix

$$K=\begin{pmatrix} P & Q\\ R & S\end{pmatrix} \tag{7}$$

where

$$P=\begin{pmatrix} g & 0\\ 0 & g\end{pmatrix} \qquad Q=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$$

$$R=\begin{pmatrix} -\partial^2H/\partial x^2 & -\partial^2H/\partial y\partial x\\ -\partial^2H/\partial x\partial y & -\partial^2H/\partial y^2\end{pmatrix} \qquad S=\begin{pmatrix} g' & 0\\ 0 & g'\end{pmatrix}.$$

Then

$$K(\rho)=\det_{1\leqslant i,j\leqslant 4}(\rho\delta_{ij}-K_{ij}) \tag{8}$$

is called the Kowalevskaya determinant. It can be seen that for a homogeneous potential of degree $2m$

$$K(\rho)=(\rho+1)(\rho-g_{\mathrm{H}})[\rho^2-\rho(2g+1)+2(g+1)^2+D_m] \tag{9a}$$

where

$$D_m=\nabla^2V(k_1,K_2) \tag{9b}$$

and $g_{\mathrm{H}}=2m/(m-1)$ is the weighted degree of the Hamiltonian. Roots of the equation $K(\rho)=0$ are called Kowalevskaya exponents. In Hamiltonian systems KE come in pairs $(\rho, g_{\mathrm{H}}-1-\rho)$. The pair $(-1, g_{\mathrm{H}})$ is always present.

By Yoshida's theorems (Yoshida 1983) if there exists at least one irrational or imaginary Kowalevskaya exponent the system is not algebraically integrable. If there exists a second invariant I of weighted degree g_I satisfying the condition that its gradient does not vanish at $x=k_1$, $y=k_2$, $P_x=-gk_1$ and $P_y=-gk_2$ then a KE $\rho=g_1$ is associated with this k_1 and k_2. It can be seen that KE are the same as resonances of the P-analysis when $p=q=g$.

2.3. Stability analysis

Using Ziglin's theorem (Ziglin 1982, 1983) Yoshida proved that the existence of an exponentially unstable straight-line periodic solution signals the non-integrability of a system (Yoshida 1986). The integrability coefficient is given by

$$\lambda_m = \nabla^2 V(c_1, c_2) - (2m-1) \tag{10}$$

where $\nabla^2 V$ is Laplacian of V and c_1 and c_2 are solutions of

$$\frac{\partial V}{\partial x}(c_1, c_2) = c_1 \qquad \frac{\partial V}{\partial y}(c_1, c_2) = c_2. \tag{11}$$

Exponential instability occurs when

$$\lambda_m < 0;\ 1 < \lambda_m < 2m-1;\ 2m+2 < \lambda_m < 6m-2; \ldots;\ j(j-1)m + j < \lambda_m < j(j+1)m - j; \ldots. \tag{12}$$

So the Hamiltonian system is non-integrable in the corresponding regions.

2.4. Restrictions on KE

The relationship between the resonance of a Painlevé singularity and the KE have been clarified by Roekaerts and Schwarz (1987). They have shown that restrictions imposed on the resonances by the extended Painlevé conjecture imply that (i) all KE associated with solutions k_1, k_2, both non-zero ($p = q = g$), must be integral multiples of $1/(m-1)$, and (ii) all KE associated with solutions k_1 and k_2 with $k_1 = 0$, $k_2 \neq 0$ ($k_1 \neq 0$, $k_2 = 0$) for $p < q = g$ ($q < p = g$) must be integral multiples of $1/2s$ where $s = 1/n(m-1)$ and n is a fixed integer specific to a particular Hamiltonian. In case (i) $\rho = r$ and in case (ii) $\rho = 1/(m-1) - (r-1)/2$.

We now combine singularity analysis with stability analysis to obtain further restrictions on KE. For homogeneous potentials it follows from (9a) and the results of Roekaerts and Schwarz (1987) that the solutions of the equation

$$\rho^2 - \left(\frac{m+1}{m-1}\right)\rho + 2\left(\frac{m}{m+1}\right)^2 + D_m = 0 \tag{13}$$

must be integral multiples of $1/(m-1)$ in case (i). Hence, for integrability, D_m must be given by

$$-D_m = [k(k-m-1) + 2m^2]/(m-1)^2 \tag{14}$$

where k is an integer. Comparing equations (6) and (11) and making use of equations (9b) and (10) we find that λ_m is directly related to D_m by

$$\lambda_m = -D_m/gg' - (2m-1). \tag{15}$$

Consequently λ_m is also restricted to a set of discrete values

$$\lambda_m = k(k-m-1)/m+1. \tag{16}$$

Expressing k modulo m by

$$k = nm + i \tag{17}$$

where n is an integer and $i = 0, 1, 2, \ldots, m-1$, we have

$$\lambda_m = j(j-1)m + j \qquad \text{for } n = j,\ i = 1\ (k = jm+1) \tag{18}$$

and

$$\lambda_m = j(j+1)m - j \qquad \text{for } n = j+1, i = 0\ (k = (j-1)m). \tag{19}$$

For $jm + 1 < k < (j+1)m$

$$j(j-1)m + j < \lambda_m < j(j+1)m - j \tag{20}$$

and hence is in the unstable region. It follows that for integrability k can assume only the values jm or $jm+1$ for arbitrary j. In other words apart from -1 and g_H the only values that the KE in case (i) can assume are 0, 1 (mod m) in units of $1/(m-1)$. The integrability coefficient λ_m then assumes only the values corresponding to boundaries separating stable and unstable regions.

In case (ii) the solution of (13) must be a multiple of $1/2s$. Hence for integrability

$$-D_m = [k/2n(k/2n - m - 1) + 2m^2]/(m-1)^2 \tag{21}$$

where k is an integer. Correspondingly the integrability coefficient is

$$\lambda_m = (k/2n)(k/2n - m - 1)/m + 1. \tag{22}$$

If $k = 2nj$, equation (22) is formally the same as (16) with $k \to j$. By repetition of the previous reasoning it will then follow that integrable cases correspond to $j = 0$, 1 (mod m). However, there can also exist other integrable cases with $k \neq 2nj$ depending on the values of n and m.

3. Integrable potentials

We have performed the Painlevé analysis and calculated KE for symmetric quartic, sextic and octic potentials with a view to identifying possible integrable cases in the light of the above results. Direct construction of the second integral of motion is also given in some cases. A generalisation of the integrable cases to potentials of arbitrary degree $2m$ is also obtained.

3.1. Quartic potentials

Consider a system with Hamiltonian

$$H = \tfrac{1}{2}(P_x^2 + P_y^2) + A(x^4 + y^4) + B(x^3y + xy^3) + Cx^2y^2 \qquad A, B, C \neq 0 \tag{23}$$

and equations of motion

$$\begin{aligned} &\dot{x} = P_x \qquad \dot{y} = P_y \\ &\dot{P}_x = -[4Ax^3 + B(3x^2y + y^3) + 2Cxy^2] \\ &\dot{P}_y = -[4Ay^3 + B(x^3 + 3xy^2) + 2Cx^2y]. \end{aligned} \tag{24}$$

To perform the Painlevé analysis we look for dominant behaviour near a singularity of the form (3). Substituting (3) in (22) give $p = q = 1$ with $b_0 = \alpha a_0$, where α can assume one of the four possible values

$$\alpha_{1,2} = \pm 1 \qquad \alpha_{1,4} = \{(4A - 2c) \pm [(4A - 2C) - 4B]^{1/2}\}/2B. \tag{25}$$

Correspondingly

$$a_0^2 = -2/(4A + 3B\alpha + 2C\alpha^2 + B\alpha^3). \tag{26}$$

Solutions of (24) can be expanded in the form

$$x(t)=\sum_{j=0}^{\infty} a_j\tau^{-1+j} \qquad y(t)=\sum_{j=0}^{\infty} b_j\tau^{-1+j}. \tag{27}$$

This is a strong P-case. Resonances are found to be $-1, 1, 2, 4$ ($\alpha=\pm1$) and sufficient arbitrary constants enter with the above type of solutions, when $C=6A$, A and B arbitrary.

To calculate KE and integrability coefficients we note that for the system (23), $g=1$, $g'=2$ and $m=2$. A solution of (6) is $k_2=\alpha k_1$ (correspondingly $c_2=\alpha c_1$ in (11)) and

$$k_1^2=-gg'c_1^2=a_0^2. \tag{28}$$

By the restrictions mentioned in § 2 the KE (in case (i) with k_1, k_2 both non-zero) can only be $1, 2, 3, \ldots$, i.e. D_2 can have values $-6, -8, -12, -18, \ldots$ and corresponding values of λ_2 are $0, 1, 3, 6, \ldots$ for any choice of solutions. For the P-case, $C=6A$ (A and B arbitrary) KE are $-1, 1, 2, 4$ for $\alpha=\pm1$ ($D_2=-6$) and $-1, -1, 4, 4$, for $\alpha=\alpha_{3,4}$ ($D_2=-12$) and the corresponding values of λ_2 are 0 and 3 respectively.

Of the possible integrable cases corresponding to the allowed values of D_2, for the P-case, we have been able to construct the following second integral of motion directly from the Poisson bracket condition $[H, I]=0$, assuming the weighted degree $\leqslant 4$:

$$I=P_xP_y+B(x^4+y^4+6x^2y^2)+4A(x^3y+xy^3). \tag{29}$$

The special case of the Hamiltonian (23) with $B=0$ has been discussed by Steeb *et al* (1985).

3.2. Sextic potentials

Consider the Hamiltonian

$$H=\tfrac{1}{2}(P_x^2+P_y^2)+A(x^6+y^6)+B(x^5y+xy^5)+C(x^4y^2+x^2y^4)+Dx^3y^3$$
$$A, B, C, D\neq 0. \tag{30}$$

Equations of motion are

$$\begin{aligned}&\dot{x}=P_x \qquad \dot{y}=P_y\\ &\dot{P}_x=-[6Ax^5+B(5x^4y+y^5)+C(4x^3y^2+2xy^4)+3Dx^2y^3]\\ &\dot{P}_y=-[6Ay^5+B(x^5+5xy^4)+C(2x^4y+4x^2y^3)+3Dx^3y^2].\end{aligned} \tag{31}$$

For this system, we have a singularity with dominant behaviour (3) with $p=q=\frac{1}{2}$ and $b_0=\alpha a_0$, where α is a root of the equation

$$B(\alpha^6-1)+(2C-6A)\alpha(\alpha^4-1)+(3D-5B)\alpha^2(\alpha^2-1)=0 \tag{32}$$

where $\alpha=\pm1$ is a root of the equation. Correspondingly

$$a_0^4=-3/[4(6A+5B\alpha+B\alpha^5+2C\alpha^4+4C\alpha^2+3D\alpha^3)]. \tag{33}$$

Solutions of (29) will be of the form

$$x(t)=\sum_{j=0}^{\infty} a_j\tau^{-1/2+j/2} \qquad \text{and} \qquad y(t)=\sum_{j=0}^{\infty} b_j\tau^{-1/2+j/2}. \tag{34}$$

This is a weak P-case. The resonances are found to be $-1, \frac{1}{2}, \frac{3}{2}, 3$ (with $\alpha=\pm1$) and a sufficient number of arbitrary constants enter in the solution when $C=15A$ and $10B=3D$ with A and B arbitrary.

For the system (30) $g=\frac{1}{2}$, $g'=\frac{3}{2}$ and $m=3$. $k_2=\alpha k_1$ is a choice of solution of (6) (correspondingly $c_2=\alpha c_1$ in (11)) and $k_1^4=-gg'c_1^4=a_0^4$. In order that the system be integrable D_3 has to be $-\frac{15}{4}, -\frac{18}{4}, -\frac{30}{4}, -\frac{39}{4}, \ldots$ and corresponding values of λ_3 are 0, 1, 5, 8, ... for any choice. For $C=15A$ and $10B=3D$ (A and B arbitrary) KE are -1, $\frac{1}{2}$, $\frac{3}{2}$, 3 ($\alpha=\pm 1$) and $\lambda_3=0$.

Looking for an integral of motion with weighted degree $\leqslant 3$ we find in the P-case

$$I=P_xP_y+B[x^6+y^6+15(x^4y^2+x^2y^4)]+A[6(x^5y+xy^5)+20x^3y^3]. \quad (35)$$

Special cases of the Hamiltonian of the form (30) with $B=D=0$ have been discussed by Graham *et al* (1985).

3.3. Octic potentials

For the Hamiltonian

$$H=\tfrac{1}{2}(P_x^2+P_y^2)+A(x^8+y^8)+B(x^7y+xy^7)+C(x^6y^2+x^2y^6)+D(x^5y^3+x^3y^5)+Ex^4y^4 \quad (36)$$

it is found that $C=28A$, $E=70A$ and $D=7B$, A and B arbitrary, is a P-case. For this system $g=\frac{1}{3}$, $g'=\frac{4}{3}$ and $m=4$. D_4 can take values $-\frac{28}{9}, -\frac{32}{9}, -\frac{56}{9}, -\frac{76}{9}, \ldots$ and corresponding values of λ_4 are 0, 1, 7, 12, 22, ... for any choice of solutions. For the P-case we have a solution for which KE are -1, $\frac{1}{3}$, $\frac{4}{3}$ and $\frac{8}{3}$ and $\lambda_4=0$ yielding an integrable case. We can also identify the following non-integrable cases: (i) $B=D=0$ (except when (a) $C=28A$, $E=70A$, (b) $C=4A$, $E=6A$ and (c) $C=E=0$); (ii) $A=B=C=D=0$; (iii) $A=C=D=E=0$, and (iv) $A=B=C=E=0$.

Searching for an integral of motion with weighted degree $\leqslant\frac{8}{3}$ we have, when

(a) $C=28A$, $E=70A$ and $D=7B$

$$I=P_xP_y+A[8(x^7y+xy^7)+56(x^5y^3+x^3y^5)] + B[x^8+y^8+28(x^6y^2+x^2y^6)+70x^4y^4] \quad (37)$$

(b) $C=4A$, $E=6A$ and $B=D=0$

$$I=P_xy-P_yx \quad (38)$$

(c) $B=C=D=E=0$

$$I=P_x^2+2x^8 \qquad \text{or} \qquad I=P_y^2+2y^8. \quad (39)$$

3.4. Generalisation

We can generalise the integrable cases to arbitrary m ($m\geqslant 2$). The general form of an integrable symmetric Hamiltonian with homogeneous potential of degree $2m$ is

$$H=\tfrac{1}{2}(P_x^2+P_y^2)+AV_m+BJ_m \quad (40)$$

and its integral of motion with a weighted degree $2m/(m-1)$ is

$$I=P_xP_y+BV_m+AJ_m \quad (41)$$

where

$$V_m = \sum_{j=0}^{m} \alpha_j x^{2m-2j} y^{2j} \tag{42}$$

$$\alpha_j = \begin{cases} \binom{2m}{2j} & \text{when } 2j \leqslant m \\ \binom{2m}{2m-2j} & \text{when } 2j > m \end{cases}$$

and

$$J_m = \sum_{j=0}^{m-1} \beta_j x^{2m-2j-1} y^{2j+1} \tag{43}$$

$$\beta_j = \begin{cases} \binom{2m}{2j+1} & \text{when } 2j+1 \leqslant m \\ \binom{2m}{2m-(2j+1)} & \text{when } 2j+1 > m. \end{cases}$$

Integrable cases (29), (35) and (37) are special cases of (41) for $m=2$, 3 and 4 respectively.

4. Conclusion

In this paper an attempt was made to combine singularity and stability analyses for a Hamiltonian system with a homogeneous potential. A new restriction on KE, which may be used as an effective tool in the search for integrable systems, has been obtained. Applying this to symmetric quartic, sextic and octic potentials we have identified possible candidates for integrability. However, it happens that the cases where we have been able to construct a second integral of motion directly are not genuinely new integrable systems. This is because potentials of the integrable form (40) can be reduced, by a rotation through an angle $\pi/4$ and scaling (Hietarinta 1987), to known integrable potentials of the form

$$V = x^n + ay^n. \tag{44}$$

It is known that the general form of integrable symmetric potnetials are $V=f(x^2+y^2)$ with integrals of motion $I = P_x y - P_y x$ and $V=f(x)+f(y)$ with integrals of motion $I = P_x^2 + 2f(x)$ or $P_y^2 + 2f(y)$ (Hietarinta 1987). Integrals of motion (38) and (39) are also special cases of these. The question whether these exhaust the integrable cases or whether there can exist an additional integral in the rest of the cases is presently under investigation.

Acknowledgment

One of the authors (JMP) is grateful to the Council of Scientific and Industrial Research, New Delhi, for the award of a Junior Research Fellowship.

References

Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
Bountis T, Segur H and Vivaldi F 1982 *Phys. Rev.* A **25** 1257
Dorizzi B, Grammaticos B and Ramani A 1983 *J. Math. Phys.* **24** 2282
Graham R, Roekaerts D and Tel T 1985 *Phys. Rev.* A **31** 3364
Grammaticos B, Dorizzi B and Ramani A 1983 *J. Math. Phys.* **24** 2289
Hietarinta J 1987 *Phys. Rep.* **147** 87
Ramani A, Dorizzi B and Grammaticos B 1982 *Phys. Rev. Lett.* **49** 1539
Roekaerts D and Schwarz F 1987 *J. Phys. A: Math. Gen.* **20** L127
Steeb W-H, Villet C M and Kunick A 1985 *J. Phys. A: Math. Gen.* **18** 3269
Yoshida H 1983 *Celest. Mech.* **31** 363, 381
—— 1984 *Celest. Mech.* **32** 73
—— 1986 *Physica* **21D** 163
Ziglin S L 1982 *Funct. Anal. Appl.* **16** 181
—— 1983 *Funct. Anal. Appl.* **17** 6